Brownian motion and Stochastic Calculus
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Assignment 9—solutions

Exercise 1

Let $W = (W_t)_{t\geq 0}$ be a 1-dimensional $(\mathbb{F},\mathbb{P})$–Brownian motion.

1) Prove that for every polynomial p on $\mathbb{R}$, the stochastic integral $\int_0^\cdot p(W_s)\mathrm{d}W_s$ is well defined. Moreover, show that it is also an $(\mathbb{F},\mathbb{P})$-martingale.

2) Show that the process $X = (X_t)_{t\geq 0}$ given by $X_t := \mathrm{e}^{\frac{1}{2}t}\cos(W_t)$, $t \geq 0$ is an $(\mathbb{F},\mathbb{P})$-martingale.

3) Let W' be another $(\mathbb{F},\mathbb{P})$–Brownian motion independent of W and ρ be an $\mathbb{F}$-adapted, measurable, process satisfying $|\rho| \leq 1$. Prove that the process B given by

$$B_t = \int_0^t \rho_s \mathrm{d}W_s + \int_0^t \sqrt{1-\rho_s^2}\mathrm{d}W'_s$$

is an $(\mathbb{F},\mathbb{P})$–Brownian motion. Moreover, compute $[B,W]$.

1) **By linearity, it suffices to check the claim for monomials of the form $p(x) = x^m$, $m \in \mathbb{N}$. Note that $p(W)$ is (left-)continuous and adapted, and hence predictable and locally bounded. Therefore, $\int_0^\cdot p(W_s)\mathrm{d}W_s$ is well-defined, and also a local martingale. Moreover, by Fubini's Theorem, for all $T \geq 0$,**

$$\begin{aligned}\mathbb{E}^\mathbb{P}\bigg[\bigg[\int_0^\cdot p(W_s)\mathrm{d}W_s\bigg]_T\bigg] = \mathbb{E}^\mathbb{P}\bigg[\int_0^T W_s^{2m}\mathrm{d}[W]_s\bigg] &= \mathbb{E}^\mathbb{P}\bigg[\int_0^T W_s^{2m}\mathrm{d}s\bigg] \\ &= \int_0^T \mathbb{E}^\mathbb{P}\big[W_s^{2m}\big]\mathrm{d}s \\ &= \mathbb{E}^\mathbb{P}[W_1^{2m}]\int_0^T s^m\mathrm{d}s < \infty.\end{aligned}$$

This proves that $\int_0^\cdot p(W_s)\mathrm{d}W_s$ is a true martingale.

2) **The function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(t,w) := \mathrm{e}^{\frac{1}{2}t}\cos w$ is C^2 and $X_t = f(t,W_t)$. Moreover**

$$\frac{\partial f}{\partial t}(t,w) = \frac{1}{2}\mathrm{e}^{\frac{1}{2}t}\cos w,\ \frac{\partial f}{\partial w}(t,w) = -\mathrm{e}^{\frac{1}{2}t}\sin w,\ \frac{\partial^2 f}{\partial w^2}(t,w) = -\mathrm{e}^{\frac{1}{2}t}\cos w.$$

Since t (viewed as a process) is of finite variation, Itô's formula yields

$$\mathrm{d}X_t = \frac{\partial f}{\partial t}(t,w)\mathrm{d}t + \frac{\partial f}{\partial w}(t,w)\mathrm{d}W_t + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(t,w)\mathrm{d}[W]_t = -\mathrm{e}^{\frac{1}{2}t}\sin W_t\mathrm{d}W_t,$$

so X is a local martingale. Since $\sup_{0\leq t\leq T}|X_t| \leq \mathrm{e}^{\frac{1}{2}T}$ for each $T \geq 0$, X is a martingale.

3) **Being adapted, left-continuous and bounded, ρ and $\sqrt{1-\rho^2}$ are such that the corresponding stochastic integrals are well-defined. Moreover, for each $t \geq 0$, using bi-linearity of $[\cdot,\cdot]$ and the fact that $[W,W'] = 0$ due to independence of W and W'**

$$[B]_t = \bigg[\int_0^\cdot \rho_s\mathrm{d}W_s\bigg]_t + \bigg[\int_0^\cdot \sqrt{1-\rho_s^2}\mathrm{d}W'_s\bigg]_t = \int_0^t \rho_s^2\mathrm{d}s + \int_0^t (1-\rho_s^2)\mathrm{d}s = t,$$

so Lévy's characterisation of Brownian motion yields that B is a Brownian motion. Finally

$$[B,W]_t = \int_0^t \rho_s \mathrm{d}[W,W]_s = \int_0^t \rho_s \mathrm{d}s.$$

Exercise 2

For any $M \in \mathcal{M}_{c,\mathrm{loc}}(\mathbb{R},\mathbb{F},\mathbb{P})$, define $M_t^\star := \sup_{0\le s\le t}|M_s|$, for $t\ge 0$. Prove that for any $t\ge 0$ and positive C, K, we have

$$\mathbb{P}\big[M_t^\star > C\big] \le \frac{4K}{C^2} + \mathbb{P}\big[[M]_t > K\big].$$

Recall that for a stopping time τ and a process $(M_t)_{t\ge0}$ the stopped process is defined by $(M_t^\tau)_{t\ge0} = (M_{\tau\wedge t})_{t\ge0}$. For $K>0$, we consider the stopping time $\sigma_K := \inf\{t>0 : [M]_t > K\}$. Since $[M]$ is continuous, we have that $[M]_t \le K$ for $t\le\sigma_K$, and therefore

$$\mathbb{E}^{\mathbb{P}}\big[[M^{\sigma_K}]_\infty\big] = \mathbb{E}^{\mathbb{P}}\big[[M]_{\sigma_K}\big] \le K.$$

Hence, $M^{\sigma_K} \in \mathcal{M}_c^2(\mathbb{R},\mathbb{F},\mathbb{P})$. We can therefore apply Tchebycheff's and Doob's inequality (and use that the constant in Doob's inequality for fixed $p>1$, denoted by C_p, is equal to $\big(\frac{p}{p-1}\big)^p$), obtaining that

$$\begin{aligned}\mathbb{P}\big[(M^{\sigma_K})_t^\star > C\big] \le \frac{\mathbb{E}^{\mathbb{P}}\big[((M^{\sigma_K})_t^\star)^2\big]}{C^2} &\le \frac{4\mathbb{E}^{\mathbb{P}}\big[(M^{\sigma_K})_t^2\big]}{C^2}\\ &= \frac{4\mathbb{E}^{\mathbb{P}}\big[[M^{\sigma_K}]_t\big]}{C^2} \le \frac{4K}{C^2}.\end{aligned}$$

To obtain the claim, we observe that

$$\{M_t^{\sigma_K} \ne M_t\} \subseteq \{\sigma_K < t\} = \{[M]_t > K\},$$

which finally implies that

$$\mathbb{P}\big[M_t^\star > C\big] = \mathbb{P}\big[M_t^\star > C,\ \sigma_K \ge t\big] + \mathbb{P}\big[M_t^\star > C,\ \sigma_K < t\big] \le \frac{4K}{C^2} + \mathbb{P}\big[[M]_t > K\big].$$

Exercise 3

Let $(B_t)_{t\ge0}$ be a Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Consider the SDE

$$\mathrm{d}X_t = \bigg(\sqrt{1+X_t^2} + \frac{1}{2}X_t\bigg)\mathrm{d}t + \sqrt{1+X_t^2}\,\mathrm{d}B_t,\ X_0 = x\in\mathbb{R}. \tag{0.1}$$

1) Show that for any $x\in\mathbb{R}$, the SDE defined in (0.1) has a unique strong solution.
2) Show that $(X_t)_{t\ge0}$ defined by $X_t := \sinh\big(\operatorname{arcsinh} x + t + B_t\big)$ is the unique solution of (0.1).

1) **We see that the SDE is of the form**

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + b(X_t)\mathrm{d}B_t,\ X_0 = x\in\mathbb{R}.$$

where

$$a(x) := \sqrt{1+x^2} + \frac{x}{2},\ \textbf{and}\ b(x) := \sqrt{1+x^2}.$$

We observe that

$$\sup_{x\in\mathbb{R}}|b'(x)| = \sup_{x\in\mathbb{R}}\bigg|\frac{x}{\sqrt{1+x^2}}\bigg| \le 1,$$

as well as

$$\sup_{x\in\mathbb{R}}|a'(x)| = \sup_{x\in\mathbb{R}}\left|\frac{x}{\sqrt{1+x^2}}+\frac{1}{2}\right| \leq \frac{3}{2}.$$

Thus, from the mean value theorem, we obtain for $K := \frac{5}{2}$ that $a(\cdot)$ and $b(\cdot)$ satisfy the Lipschitz condition

$$|a(y)-a(z)|+|b(y)-b(z)| \leq K|y-z|,\ (y,z)\in\mathbb{R}^2.$$

Moreover, we observe that for any $x \in \mathbb{R}$

$$\left|\sqrt{1+x^2}+\frac{x}{2}\right| \leq \left|1+|x|+\frac{x}{2}\right| \leq \frac{3}{2}\big(1+|x|\big),\ \big|\sqrt{1+x^2}\big| \qquad\qquad \leq 1+|x|.$$

Thus we get for any $x \in \mathbb{R}$ the existence of a unique strong solution directly from the lecture notes.

2) **We consider the function $f(x) :=$ arsinh$(x) \in C^2$ (i.e. the inverse function of the hyperbolic sine). Thus, we obtain that**

$$f'(x) = \frac{1}{\sqrt{1+x^2}},\ \textbf{and } f''(x) = -\frac{x}{(1+x^2)^{3/2}}.$$

Thus, applying Itô's formula to $Y_t := f(X_t)$, we obtain that

$$\mathrm{d}Y_t = \mathrm{d}t + \mathrm{d}B_t,\ Y_0 = \textbf{arsinh}(x),$$

which implies that

$$X_t = \sinh(Y_t) = \sinh\big(\textbf{arsinh}(x+t+B_t)\big),\ t \geq 0.$$

Exercise 4

Let B be a Brownian motion defined on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ satisfying the usual conditions, and let us fix three constants $(a,b,\sigma)\in(0,+\infty)^3$, and an initial value $r_0\in\mathbb{R}$. An Ornstein–Uhlenbeck process r satisfies the following SDE

$$r_t = r_0 + \int_0^t (a-br_s)\mathrm{d}s + \sigma B_t,\ t\geq 0.$$

1) Show that

$$r_s = \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b} + \int_t^s \mathrm{e}^{-b(s-u)}\sigma\mathrm{d}B_u,\ 0\leq t\leq s.$$

2) Deduce that the $\mathbb{P}$-distribution of r_s knowing $\mathcal{F}_t$ is Gaussian with mean

$$m(t,s) := \mathbb{E}^{\mathbb{P}}\big[r_s\big|\mathcal{F}_t^{B,\mathbb{P}}\big] = \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b},$$

and variance

$$v(t,s) := \mathbb{V}\mathrm{ar}^{\mathbb{P}}\big[r_s\big|\mathcal{F}_t^{B,\mathbb{P}}\big] = \sigma^2\int_t^s \mathrm{e}^{-2b(s-u)}du = \frac{\sigma^2}{2b}\big(1-\mathrm{e}^{-2b(s-t)}\big).$$

3) Prove the following stochastic Fubini theorem.

Lemma 0.1. *Let b and σ be two $\mathbb{R}$-valued measurable and $\mathbb{F}$-adapted processes such that for any $t\geq 0$*

$$\int_0^t \big(|b_s|+|\sigma_s|^2\big)\mathrm{d}s < +\infty.$$

We have for any $t \geq 0$

$$\int_0^t b_s\Big(\int_0^s \sigma_u\mathrm{d}B_u\Big)\mathrm{d}s = \Big(\int_0^t \sigma_u\mathrm{d}B_u\Big)\Big(\int_0^t b_s\mathrm{d}s\Big) - \int_0^t \sigma_u\Big(\int_0^u b_s\mathrm{d}s\Big)\mathrm{d}B_u.$$

Deduce from this that

$$\int_t^s r_u \mathrm{d}u = \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t) + \sigma\int_t^s \frac{1-\mathrm{e}^{-b(s-u)}}{b}\mathrm{d}B_u,$$

and that the distribution of $\int_t^s r_u \mathrm{d}u$, conditionally on $\mathcal{F}_t$, is Gaussian with

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^s r_u \mathrm{d}u \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] = \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t),$$

and

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\int_t^s r_u \mathrm{d}u \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] = \frac{\sigma^2}{b^2}\left(s-t-\frac{2(1-\mathrm{e}^{-b(s-t)})}{b} + \frac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right).$$

4) Finally, prove that the joint distribution, knowing $\mathcal{F}_t$, of the vector $(r_s, \int_t^s r_u \mathrm{d}u)$ is still Gaussian with mean given by the vector

$$\begin{pmatrix} \mathrm{e}^{-b(s-t)}r_t + a\dfrac{1-\mathrm{e}^{-b(s-t)}}{b} \\ \dfrac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \dfrac{a}{b}\right) + \dfrac{a}{b}(s-t) \end{pmatrix},$$

and covariance matrix

$$\begin{pmatrix} \dfrac{\sigma^2}{2b}\left(1-\mathrm{e}^{-2b(s-t)}\right) & \sigma^2\left(\dfrac{1-\mathrm{e}^{-b(s-t)}}{b^2} - \dfrac{1-\mathrm{e}^{-2b(s-t)}}{2b^2}\right) \\ \sigma^2\left(\dfrac{1-\mathrm{e}^{-b(s-t)}}{b^2} - \dfrac{1-\mathrm{e}^{-2b(s-t)}}{2b^2}\right) & \dfrac{\sigma^2}{b^2}\left(s-t-\dfrac{2(1-\mathrm{e}^{-b(s-t)})}{b} + \dfrac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right) \end{pmatrix}.$$

1) **If we define the process X by $X_t := \mathrm{e}^{bt}r_t$, for any $t \geq 0$, Itô's formula shows that it satisfies**

$$X_t = r_0 + \int_0^t \mathrm{e}^{bs}a\mathrm{d}s + \int_0^t \sigma\mathrm{e}^{bs}\mathrm{d}B_s,\ t \geq 0.$$

This implies that for any $s \geq t \geq 0$

$$X_s - X_t = a\int_t^s \mathrm{e}^{bu}\mathrm{d}u + \sigma\int_t^s \mathrm{e}^{bu}\mathrm{d}B_u,$$

and replacing X_s and X_t by their values

$$r_s = \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b} + \int_t^s \mathrm{e}^{-b(s-u)}\sigma\mathrm{d}B_u.$$

2) **Hence, r_s, given $\mathcal{F}_t^{B,\mathbb{P}}$, is given as a deterministic function, plus a stochastic integral of a deterministic function. We know by the lecture notes that the distribution of such a stochastic integral is Gaussian. We then have**

$$m(t,s) := \mathbb{E}^{\mathbb{P}}\left[r_s \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] = \mathrm{e}^{-b(s-t)}r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b},$$

and

$$v(t,s) := \mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[r_s \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] = \sigma^2\int_t^s \mathrm{e}^{-2b(s-u)}du = \frac{\sigma^2}{2b}\left(1-\mathrm{e}^{-2b(s-t)}\right).$$

3) **We apply Itô's formula for products to obtain**

$$\left(\int_0^t \sigma_u \mathrm{d}B_u\right)\left(\int_0^t b_s \mathrm{d}s\right) = \int_0^t \sigma_u\left(\int_0^u b_s \mathrm{d}s\right)\mathrm{d}B_u + \int_0^t b_s\left(\int_0^s \sigma_u \mathrm{d}B_u\right)\mathrm{d}s,$$

from which the result is immediate.

We deduce then that

$$\int_t^s r_u \mathrm{d}u = \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t) + \sigma \int_t^s \frac{1-\mathrm{e}^{-b(s-u)}}{b}\mathrm{d}B_u.$$

and in particular that the distribution of $\int_t^s r_u \mathrm{d}u$, conditionally on $\mathcal{F}_t$, is Gaussian with

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^s r_u \mathrm{d}u \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] = \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t),$$

and

$$\mathbb{V}\mathrm{ar}^{\mathbb{P}}\left[\int_t^s r_u \mathrm{d}u \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] = \frac{\sigma^2}{b^2}\left(s - t - \frac{2(1-\mathrm{e}^{-b(s-t)})}{b} + \frac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right).$$

4) **Concerning the joint distribution, we have for any** $(\lambda, \rho) \in \mathbb{R}^2$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{i\lambda r_s + i\rho \int_t^s r_u \mathrm{d}u} \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] &= \exp\left(i\lambda\left(\mathrm{e}^{-b(s-t)} r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b}\right) + i\rho\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t)\right)\right) \\
&\quad \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(i\sigma \int_t^s \frac{\rho + (\lambda b - \rho)\mathrm{e}^{-b(s-u)}}{b}\mathrm{d}B_u\right) \middle| \mathcal{F}_t^{B,\mathbb{P}}\right] \\
&= \exp\left(i\lambda\left(\mathrm{e}^{-b(s-t)} r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b}\right) + i\rho\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t)\right)\right) \\
&\quad \times \exp\left(-\frac{\sigma^2}{2b^2}\int_t^s \left(\rho + (\lambda b - \rho)\mathrm{e}^{-b(s-u)}\right)^2 \mathrm{d}u\right) \\
&= \exp\left(i\lambda\left(\mathrm{e}^{-b(s-t)} r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b}\right) + i\rho\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t)\right)\right) \\
&\quad \times \exp\left(-\frac{\sigma^2}{2b^2}\left(\rho^2(s-t) + 2\rho(\lambda b - \rho)\frac{1-\mathrm{e}^{-b(s-t)}}{b} + (\lambda b - \rho)^2 \frac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right)\right),
\end{aligned}$$

which is the characteristic function of bi-dimensional Gaussian random variable with mean given by the vector

$$\begin{pmatrix} \mathrm{e}^{-b(s-t)} r_t + a\frac{1-\mathrm{e}^{-b(s-t)}}{b} \\ \frac{1-\mathrm{e}^{-b(s-t)}}{b}\left(r_t - \frac{a}{b}\right) + \frac{a}{b}(s-t) \end{pmatrix},$$

and covariance matrix

$$\begin{pmatrix} \frac{\sigma^2}{2b}\left(1-\mathrm{e}^{-2b(s-t)}\right) & \sigma^2\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b^2} - \frac{1-\mathrm{e}^{-2b(s-t)}}{2b^2}\right) \\ \sigma^2\left(\frac{1-\mathrm{e}^{-b(s-t)}}{b^2} - \frac{1-\mathrm{e}^{-2b(s-t)}}{2b^2}\right) & \frac{\sigma^2}{b^2}\left(s - t - \frac{2(1-\mathrm{e}^{-b(s-t)})}{b} + \frac{1-\mathrm{e}^{-2b(s-t)}}{2b}\right) \end{pmatrix}.$$

Exercise 5

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. Assume that the filtration $\mathbb{F}$ is generated by the Brownian motion W. Consider the Tanaka SDE

$$\mathrm{d}X_t = \mathrm{sgn}(X_t)\mathrm{d}W_t,\ X_0 = 0,$$

where $\mathrm{sgn}(x)$ denotes the sign function, i.e., $\mathrm{sgn}(x) := 1$ if $x > 0$, and $\mathrm{sgn}(x) = -1$ if $x \leq 0$.

1) Show that the Tanaka SDE has no strong solution.

Hint:

- Assume there exists a strong solution and derive a contradiction.
- You can use the following result (Tanaka's formula): let X be a continuous semimartingale. There exists a continuous, non-decreasing adapted process $(L_t)_{t\geq 0}$ such that

$$|X_t| - |X_0| = \int_0^t sgn(X_s)dX_s + L_t,\ t \geq 0.$$

  Moreover, it can be shown that L is $\mathbb{F}^{|X|}$-adapted.

2) Show that the SDE admits a weak solution.

1) **By contradiction, suppose that X has a strong solution. Since X is $\mathbb{F}$ adapted we have $\mathbb{F}^X \subseteq \mathbb{F} = \mathbb{F}^W$. Moreover, since $\mathrm{sgn}(X_\cdot)$ is adapted and left-continuous, X is a continuous local $(\mathbb{F}, \mathbb{P})$-martingale null at 0 with**

$$[X]_t = \int_0^t (\mathrm{sgn}(X_s))^2 \mathrm{d}[W]_t = t.$$

Therefore, by Lévy's theorem, X is even an $(\mathbb{F}, \mathbb{P})$–Brownian motion. By definition, we have

$$W_t = \int_0^t (\mathrm{sgn}(X_s))^2 \mathrm{d}W_s = \int_0^t \mathrm{sgn}(X_s)\mathrm{d}X_s.$$

Using Tanaka's formula we see that W is adapted to $\mathbb{F}^{|X|}$. Hence, we have $\mathbb{F}^X \subseteq \mathbb{F} = \mathbb{F}^W \subseteq \mathbb{F}^{|X|}$ which is clearly a contradiction.

2) **To find a weak solution let $\mathbb{Q}$ be the Wiener measure on the path space $\Omega = C[0, \infty)$ and X be the coordinate process such that X is an $(\mathbb{F}^X, \mathbb{Q})$–Brownian motion. Moreover, let $\mathbb{F}$ be the (augmented) canonical filtration and define W as**

$$W := \int_0^\cdot \mathrm{sgn}(X_s)\mathrm{d}X_s.$$

As before, using Lévy's theorem W is an $(\mathbb{F}, \mathbb{Q})$–Brownian motion. Therefore

$$\mathrm{sgn}(X_t)\mathrm{d}W_t = (\mathrm{sgn}(X_t))^2 \mathrm{d}X_t = \mathrm{d}X_t.$$